Chapter 6

The Time Value of Money: Annuities and Other Topics

Outline

- Learning Objectives
- Principles Used in This Chapter
  1. Annuities
  2. Perpetuities
  3. Complex Cash Flow Streams

Learning Objectives

1. Distinguish between an ordinary annuity and an annuity due, and calculate present and future values of each.
2. Calculate the present value of a level perpetuity and a growing perpetuity.
   - Need this for stock valuation
3. Calculate the present and future value of complex cash flow streams.
   - Need this for bond valuation, capital budgeting

Principles Used in This Chapter

- Principle 1: Money Has a Time Value.
  - This chapter applies the time value of money concepts to annuities, perpetuities and complex cash flows.
- Principle 3: Cash Flows Are the Source of Value.
  - This chapter introduces the idea that principle 1 and principle 3 will be combined to value stocks, bonds, and investment proposals.

6.1 Annuities

Ordinary Annuities

- An **annuity** is a series of *equal dollar* payments that are made at the end of equidistant points in time such as monthly, quarterly, or annually over a *finite period* of time.
- If payments are made at the **end** of each period, the annuity is referred to as **ordinary annuity**.

Ordinary Annuities

- Example 6.1 How much money will you accumulate by the end of year 10 if you deposit $3,000 each for the next ten years in a savings account that earns 5% per year?
- We can determine the answer by using the equation for computing the future value of an ordinary annuity.

Future Value of an Ordinary Annuity

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- FV_n = FV of annuity at the end of *n*th period.
- PMT = annuity payment deposited or received at the end of each period
- i = interest rate per period
- n = number of periods for which annuity will last

Future Value of an Ordinary Annuity

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- FV = $3000 $\{[\ (1+.05)^{10} - 1] \div (.05)\}$
  = $3,000 { [0.63] ÷ (.05) }
  = $3,000 {12.58}
  = **$37,740**

Future Value of an Ordinary Annuity

- Using a Financial Calculator
- ***Enter***
  - N=10
  - 1/y = 5.0
  - PV = 0
  - PMT = -3000
  - FV = **$37,733.67**

Solving for PMT in an Ordinary Annuity

- Instead of figuring out how much money you will accumulate (i.e. FV), you may like to know how much you need to save each period (i.e. PMT) in order to accumulate a certain amount at the end of n years.
- In this case, we know the values of n, i, and FV_n in equation 6-1c and we need to determine the value of PMT.

Solving for PMT in an Ordinary Annuity

- Example 6.2: Suppose you would like to have $25,000 saved 6 years from now to pay towards your down payment on a new house. If you are going to make equal annual end-of-year payments to an investment account that pays 7 per cent, how big do these annual payments need to be?

Solving for PMT in an Ordinary Annuity

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- Here we know, FV_n = \$25,000; n = 6; and i=7% and we need to determine PMT.

Solving for PMT in an Ordinary Annuity

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- \$25,000 = **PMT** {[$(1+.07)^6 - 1$] ÷ (.07)}
  = PMT{ [.50] ÷ (.07) }
  = PMT {7.153}

\$25,000 ÷ 7.153 = **PMT** = **\$3,495.03**

Solving for PMT in an Ordinary Annuity

- Using a Financial Calculator.
- ***Enter***
  - N=6
  - 1/y = 7
  - PV = 0
  - FV = 25000
  - PMT = -**3,494.89**

Checkpoint 6.1
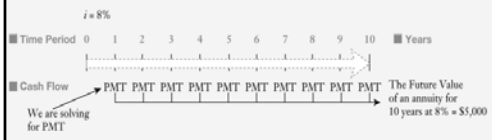
Solving for an Ordinary Annuity Payment

How much must you deposit in a savings account earning 8% annual interest in order to accumulate $5,000 at the end of 10 years? Let's solve this problem using the mathematical formulas and a financial calculator.

Checkpoint 6.1

STEP 1: Picture the problem

We can use a timeline to identify the annual payments earning 8% that must be made in order to accumulate $5,000 at the end of 10 years as follows:



STEP 2: Decide on a solution strategy

This is a future value of an annuity problem where we know the *n*, *i*, *FV*; and we are solving for *PMT* (*PV* is zero because there is no cash flow at time period 0). We'll use Equation (6–1c) to solve the problem.

Checkpoint 6.1

STEP 3: Solution

Using the Mathematical Formulas. Substituting these example values in Equation (6–1c), we find

$$\$5{,}000 = PMT\left[\frac{(1 + .08)^{10} - 1}{.08}\right]$$

$$\$5{,}000 = PMT(14.4866)$$

$$PMT = \$5{,}000 \div 14.4866 = \$345.15$$

Thus, you must deposit $345.15 in the bank at the end of each year for ten years at 8 percent interest to accumulate $5,000.

Using a Financial Calculator.

Enter	10	8.0	0		5,000
	N	I/Y	PV	PMT	FV
Solve for				-345.15	

Using an Excel Spreadsheet.

=PMT(rate,nper,pv,fv) or with values entered =PMT(.08,10,0,5000)

Checkpoint 6.1

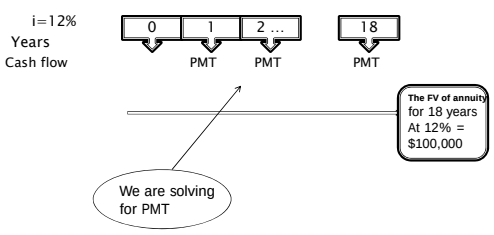
STEP 4: Analyze

Notice that in a future value of an ordinary annuity problem, the last payment actually occurs at the time the future value occurs. In this case, the last payment occurs at the end of year 10, and the end of year 10 is when you want the future value of the annuity to equal $5,000. In effect, the final payment does not have a chance to earn any interest.

Checkpoint 6.1: *Check Yourself*

If you can earn 12 percent on your investments, and you would like to accumulate $100,000 for your child's education at the end of 18 years, how much must you invest annually to reach your goal?

Step 1: Picture the Problem



Step 2: Decide on a Solution Strategy

- This is a future value of an annuity problem where we know the n, i, FV and we are solving for PMT.
- We will use equation 6-1c to solve the problem.

Step 3: Solution

- Using the Mathematical Formula

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- $100,000 = **PMT** {[$(1+.12)^{18} - 1$] ÷ (.12)}
  = PMT{ [6.69] ÷ (.12) }
  = PMT {55.75}

$100,000 ÷ 55.75 = **PMT** = **$1,793.73**

Step 3: Solution

- Using a Financial Calculator.
- ***Enter***
  - N=18
  - 1/y = 12.0
  - PV = 0
  - FV = 100000
  - PMT = -**1,793.73**

Step 4: Analyze

- If we contribute $1,793.73 every year for 18 years, we should be able to reach our goal of accumulating $100,000 if we earn a 12% return on our investments.
- Note the last payment of $1,793.73 occurs at the end of year 18. In effect, the final payment does not have a chance to earn any interest.

Solving for Interest Rate in an Ordinary Annuity

- You can also solve for "interest rate" you must earn on your investment that will allow your savings to grow to a certain amount of money by a future date.
- In this case, we know the values of n, PMT, and FV_n in equation 6-1c and we need to determine the value of i.

Solving for Interest Rate in an Ordinary Annuity

- Example 6.3: In 20 years, you are hoping to have saved $100,000 towards your child's college education. If you are able to save $2,500 at the end of each year for the next 20 years, what rate of return must you earn on your investments in order to achieve your goal?

Solving for Interest Rate in an Ordinary Annuity

- Using the Mathematical Formula

$$FV_n = PMT\left[\frac{(1+i)^n - 1}{i}\right]$$

- $100,000 = $2,500{[(1+i)^20 – 1] ÷ (i)}]
- 40 = {[(1+i)^20 – 1] ÷ (i)}
- The only way to solve for "i" mathematically is by trial and error.
- That is why we solve using a calculator

Solving for Interest Rate in an Ordinary Annuity

- Using a Financial Calculator
- *Enter*
  - N = 20
  - PMT = -$2,500
  - FV = $100,000
  - PV = $0
  - i = **6.77**

Solving for the Number of Periods in an Ordinary Annuity

- You may want to calculate the *number of periods* it will take for an annuity to reach a certain future value, given interest rate.
- It is easier to solve for number of periods using financial calculator or excel, rather than mathematical formula.

Solving for the Number of Periods in an Ordinary Annuity

- Example 6.4: Suppose you are investing $6,000 at the end of each year in an account that pays 5%. How long will it take before the account is worth $50,000?

Solving for the Number of Periods in an Ordinary Annuity

- Using a Financial Calculator
- *Enter*
  - I/y = 5.0
  - PV = 0
  - PMT = -6,000
  - FV = 50,000
  - N = **7.14**

Present Value of an Ordinary Annuity

- The present value of an ordinary annuity measures the value today of a stream of cash flows occurring in the future.

Present Value of an Ordinary Annuity

- For example, we will compute the PV of ordinary annuity if we wish to answer the question:
- What is the value today or lump sum equivalent of receiving $3,000 every year for the next 30 years if the interest rate is 5%?

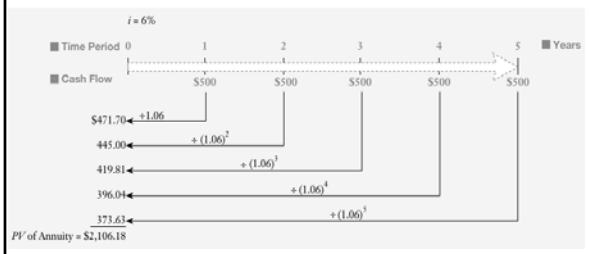
Present Value of an Ordinary Annuity

- Figure 6-2 shows the lump sum equivalent ($2,106.18) of receiving $500 per year for the next five years at an interest rate of 6%.

Figure 6.2

Timeline of a Five-Year, $500 Annuity Discounted Back to the Present at 6 Percent

To find the present value of an annuity, discount each cash flow back to the present separately and then add them. In this example, we simply add up the present value of five future cash flows of $500 each to find a present value of $2,106.18.



Present Value of an Ordinary Annuity

$$\text{Present Value} = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right]$$

- PMT = annuity payment deposited or received at the end of each period.
- i = discount rate (or interest rate) on a per period basis.
- n = number of periods for which the annuity will last.

Present Value of an Ordinary Annuity

$$\text{Present Value} = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right]$$

- **It is important that "n" and "i" match.**
- If periods are expressed in terms of number of monthly payments, the interest rate must be expressed in terms of the interest rate per month.

Checkpoint 6.2

The Present Value of an Ordinary Annuity

Your grandmother has offered to give you $1,000 per year for the next 10 years. What is the present value of this 10-year, $1,000 annuity discounted back to the present at 5 percent? Let's solve this using the mathematical formula, a financial calculator, and an Excel spreadsheet.

Checkpoint 6.2

STEP 1: Picture the problem

We can use a timeline to identify the cash flows from the investment as follows:



STEP 2: Decide on a solution strategy

In this case we are trying to determine the present value of an annuity, and we know the number of years, the dollar value that is received at the end of each year, and the number of years the annuity lasts. We also know that the discount rate is 5 percent. We can use Equation (6–2b) to solve this problem.

Checkpoint 6.2

STEP 3: Solution

Using the Mathematical Formulas. Substituting these example values in Equation (6–2b), we find that

$$PV = \$1{,}000\left[\frac{1-\frac{1}{(1+.05)^{10}}}{.05}\right] = \$1{,}000[(1-.6139)/.05] = \$1{,}000(7.722) = \$7{,}721.73$$

Using a Financial Calculator.

Enter	10	5.0		1,000	0
	N	I/Y	PV	PMT	FV
Solve for			-7,721.73		

Using an Excel Spreadsheet.

=PV(rate,nper,pmt,fv) or with values entered = PV(0.05,10,1000,0)

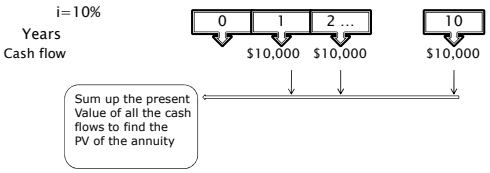
STEP 4: Analyze

We will see this formula at work a bit later when we look at the value of a bond. When you buy a bond, you get the same amount of interest every year on either an annual or semi-annual basis, and then at maturity you get the repayment of the bond's principal. Part of calculating the value of a bond involves calculating the present value of the bond's interest payments, which is an annuity.

Checkpoint 6.2: *Check Yourself*

What is the present value of an annuity of $10,000 to be received at the end of each year for 10 years given a 10 percent discount rate?

Step 1: Picture the Problem



Step 2: Decide on a Solution Strategy

- In this case we are trying to determine the present value of an annuity. We know the number of years (n), discount rate (i), dollar value received at the end of each year (PMT).
- We can use equation 6-2b to solve this problem.

Step 3: Solution

- Using the Mathematical Formula

$$\text{Present Value} = PMT\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right]$$

- PV = \$10,000 {$\left[1-(1/(1.10)^{10}\right] \div (.10)$}
  = \$10,000 {[0.6145] ÷ (.10)}
  = \$10,000 {6.145)
  = **\$ 61,445**

Step 3: Solution

- Using a Financial Calculator
- *Enter*
  - N = 10
  - I/y = 10.0
  - PMT = -10,000
  - FV = 0
  - PV = **61,445.67**

Step 4: Analyze

- A lump sum or one time payment today of $61,446 is equivalent to receiving $10,000 every year for 10 years given a 10 percent discount rate.

Amortized Loans

- An **amortized loan** is a loan paid off in equal payments – consequently, the loan payments are an annuity.
- Examples: Home mortgage loans, Auto loans

Amortized Loans

- In an amortized loan,
  - the *present value* can be thought of as the amount borrowed,
  - *n* is the number of periods the loan lasts for,
  - *i* is the interest rate per period,
  - *future value* takes on zero because the loan will be paid of after n periods, and
  - *payment* is the loan payment that is made.

Amortized Loans

- Example 6.5 Suppose you plan to get a $9,000 loan from a furniture dealer at 18% annual interest with annual payments that you will pay off in over five years.
- What will your annual payments be on this loan?

Amortized Loans

- Using a Financial Calculator
- ***Enter***
  - N = 5
  - i/y = 18.0
  - PV = 9000
  - FV = 0
  - PMT = **-$2,878.00**

The Loan Amortization Schedule

Year	Amount Owed on Principal at the Beginning of the Year (1)	Annuity Payment (2)	Interest Portion of the Annuity (3) = (1) × 18%	Repayment of the Principal Portion of the Annuity (4) = (2) −(3)	Outstanding Loan Balance at Year end, After the Annuity Payment (5) =(1) − (4)
1	$9,000	$2,878	$1,620.00	$1,258.00	$7,742.00
2	$7,742	$2,878	$1,393.56	$1,484.44	$6,257.56
3	$6257.56	$2,878	$1,126.36	$1,751.64	$4,505.92
4	$4,505.92	$2,878	$811.07	$2,066.93	$2,438.98
5	$2,438.98	$2,878	$439.02	$2,438.98	$0.00

The Loan Amortization Schedule

- We can observe the following from the table:
  - Size of each payment remains the same.
  - However, Interest payment declines each year as the amount owed declines and more of the principal is repaid.

Amortized Loans with Monthly Payments

- Many loans such as auto and home loans require monthly payments.
- This requires converting *n* to number of months and computing the monthly interest rate.

Amortized Loans with Monthly Payments

- Example 6.6 You have just found the perfect home. However, in order to buy it, you will need to take out a $300,000, 30-year mortgage at an annual rate of 6 percent.
- What will your monthly mortgage payments be?

Amortized Loans with Monthly Payments

- Mathematical Formula

$$PV = PMT\left[\frac{1 - \frac{1}{(1 + \text{annual interest rate}/m)^{\text{number of years} \times m}}}{\text{annual interest rate}/m}\right]$$

- Here annual interest rate = .06, number of years = 30, m=12, PV = $300,000

Amortized Loans with Monthly Payments

◦ $300,000= PMT $\left(\frac{1 - 1/(1+.06/12)^{360}}{.06/12}\right)$

$300,000 = PMT [166.79]

$300,000 ÷ 166.79 = **$1798.67**

Amortized Loans with Monthly Payments

- Using a Financial Calculator
- ***Enter***
  - N=360
  - I/y = .5
  - PV = 300000
  - FV = 0
  - PMT = **-1798.65**

Checkpoint 6.3

Determining the Outstanding Balance of a Loan

Let's say that exactly ten years ago you took out a $200,000, 30-year mortgage with an annual interest rate of 9 percent and monthly payments of $1,609.25.

But since you took out that loan, interest rates have dropped. You now have the opportunity to refinance your loan at an annual rate of 7 percent over 20 years.

You need to know what the outstanding balance on your current loan is so you can take out a lower-interest-rate loan and pay it off.

If you just made the 120th payment and have 240 payments remaining, what's your current loan balance?

Checkpoint 6.3

STEP 1: Picture the problem

Since we are trying to determine how much you still owe on your loan, we need to determine the present value of your remaining payments. In this case, because we are dealing with a 30-year loan, with 240 remaining monthly payments, it's a bit difficult to draw a timeline that shows all the monthly cash flows. Still, we can mentally visualize the problem which involves calculating the present value of 240 payments of $1,609.25 using a discount rate of 9%/12.

STEP 2: Decide on a solution strategy

Initially you took out a $200,000, 30-year mortgage with an interest rate of 9 percent, and monthly payments of $1,609.25. Since you have made 10 years worth of payments—that's 120 monthly payments—there are only 240 payments left before your mortgage will be totally paid off. We know that the outstanding balance is the present value of all the future monthly payments. To find the present value of these future monthly payments, we'll use Equation (6–2c).

Checkpoint 6.3

STEP 3: Solve

Using the Mathematical Formulas.

Using Equation (6–2c), we'll solve for the present value of the remaining monthly payment. To find n, we multiply the number of years left until the mortgage is paid off (20) times the number of months in a year (12). Thus n becomes 240. The future value will be equal to zero because the loan will be fully paid off in 20 years. The payment will be $1,609.25 as given above. In effect, the present value of the payments you still need to make is how much you still owe.

$$PV = PMT\left[\frac{1-\frac{1}{(1+\text{annual rate of interest}/m)^{\text{years}\times m}}}{\text{annual rate of interest}/m}\right] \quad (6\text{–}2c)$$

where m = number of times compounding occurs per year and # years is the number of years.
Substituting annual interest rate = .09, number of years = 20, m = 12, and PMT = $1,609.25 into Equation (6–2c), we get,

$$PV = \$1{,}609.25\left[\frac{1-\frac{1}{(1+.09/12)^{20\times 12}}}{.09/12}\right]$$

$$PV = \$1{,}609.25(111.145)$$

$$PV = \$178{,}860.02$$

Using a Financial Calculator.

Enter	240	9.0÷12		-1,609.25	0
	N	I/Y	PV	PMT	FV
Solve for			178,860.02		

Using an Excel Spreadsheet.

=FV(rate,nper,pmt,fv) or with values entered =FV((9/12)%,240,-1609.25,0)

Checkpoint 6.3

STEP 4: Analyze

To solve this problem, we began with our monthly payments. Then we determined what the present value was of the remaining payments—this is how much you still owe. Thus, after making 10 years of monthly payments on your $200,000 mortgage that originally had a maturity of 30 years and carries a 9% annual rate of interest with monthly payments of $1,609.25, you still owe $178,860.02.

The logic behind what was done here is that the amount you owe on a loan should be equal to the present value of the remaining loan payments. However, if interest rates dropped and you decided to refinance your mortgage, you'd find that there are some real costs associated with refinancing that we haven't touched on here. For example, there is an application fee, an appraisal fee, legal and title search fees, an origination fee for processing the loan, and a prepayment penalty, all adding to the cost of refinancing. Once you decide on a mortgage refinancing lender, make sure that you get all of your mortgage refinancing terms written down on paper.

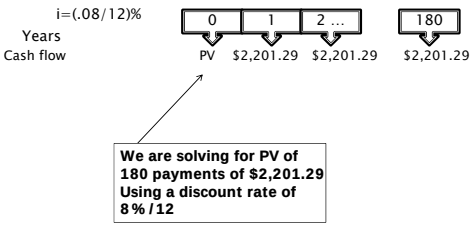
Checkpoint 6.3: *Check Yourself*

Let's assume you took out a $300,000, 30-year mortgage with an annual interest rate of 8%, and monthly payment of $2,201.29.

Since you have made 15 years worth of payments, there are 180 monthly payments left before your mortgage will be totally paid off.

How much do you still owe on your mortgage?

Step 1: Picture the Problem



We are solving for PV of 180 payments of $2,201.29 Using a discount rate of 8%/12

Step 2: Decide on a Solution Strategy

- You took out a 30-year mortgage of $300,000 with an interest rate of 8% and monthly payment of $2,201.29. Since you have made payments for 15-years (or 180 months), there are 180 payments left before the mortgage will be fully paid off.

Step 2: Decide on a Solution Strategy

- The outstanding balance on the loan at anytime is equal to the present value of all the future monthly payments.
- Here we will use equation 6-2c to determine the present value of future payments for the remaining 15-years or 180 months.

Step 3: Solve

- Using Mathematical Formula

$$PV = PMT\left[\frac{1 - \frac{1}{(1 + \text{annual interest rate}/m)^{\text{number of years} \times m}}}{\text{annual interest rate}/m}\right]$$

- Here annual interest rate = .09; number of years =15, m = 12, PMT = $2,201.29

Solve

- PV = $2,201.29 $\left(\frac{1 - 1/(1+.08/12)^{180}}{.08/12}\right)$
  = $2,201.29 [104.64]
  = **$230,344.95**

Solve

- Using a Financial Calculator
- ***Enter***
  - N = 180
  - 1/y =8/12
  - PMT = -2201.29
  - FV = 0
  - PV = **$230,344.29**

Step 4: Analyze

- The amount you owe equals the present value of the remaining payments.
- Here we see that even after making payments for 15-years, you still owe around $230,344 on the original loan of $300,000.
- Thus, most of the payment during the initial years goes towards the interest rather than the principal.

Annuities Due

- **Annuity due** is an annuity in which all the cash flows occur at the beginning of the period.
  - Rent payments on apartments are typically annuity due as rent is paid at the beginning of the month.
  - Premium payments on insurance policies are typically annuity due since they are paid at the beginning of the month or beginning of the year.

Annuities Due: Future Value

- Computation of future value of an annuity due requires compounding the cash flows for one additional period, beyond an ordinary annuity.

$$FV_n(\text{annuity due}) = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

Annuities Due: Future Value

- Recall Example 6.1 where we calculated the future value of 10-year ordinary annuity of $3,000 earning 5 per cent to be $37,734.
- What will be the future value if the deposits of $3,000 were made at the beginning of the year i.e. the cash flows were annuity due?

Annuities Due: Future Value

$$FV_n(\text{annuity due}) = PMT\left[\frac{(1+i)^n - 1}{i}\right](1+i)$$

- FV = $3000 {[$(1+.05)^{10} - 1$] ÷ (.05)} (1.05)
  = $3,000 { [0.63] ÷ (.05) } (1.05)
  = $3,000 {12.58}(1.05)
  = **$39,620**

Annuities Due: Present Value

- Since with annuity due, each cash flow is received one year earlier, its present value will be discounted back for one less period.

$$PV(\text{annuity due}) = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right](1+i)$$

Annuities Due: Present Value

- Recall checkpoint 6.2 Check yourself problem where we computed the PV of 10-year ordinary annuity of $10,000 at a 10 percent discount rate to be equal to $61,446.
- What will be the present value if $10,000 is received at the beginning of each year i.e. the cash flows were annuity due?

Annuities Due: Present Value

$$PV(\text{annuity due}) = PMT\left[\frac{1 - \frac{1}{(1+i)^n}}{i}\right](1+i)$$

- PVAD = $10,000 {[1-(1/(1.10)^{10}] ÷ (.10)} (1.1)
  = $10,000 {[0.6144] ÷ (.10)}(1.1)
  = $10,000 {6.144) (1.1)
  = **$ 67,590**

Annuities Due

- The examples illustrate that both the future value and present value of an annuity due are larger than that of an ordinary annuity because, in each case, all payments are received or paid earlier.

6.2 Perpetuities

Perpetuities

- A **perpetuity** is an annuity that continues forever or has no maturity. For example, a dividend stream on a share of preferred stock. There are two basic types of perpetuities:
  - **Level perpetuity** in which the payments are constant rate from period to period.
  - **Growing perpetuity** in which cash flows grow at a constant rate, g, from period to period.

Present Value of a Level Perpetuity

$$PV = \frac{PMT}{i}$$

- PV = the present value of a level perpetuity
- PMT = the constant dollar amount provided by the perpetuity
- i = the interest (or discount) rate per period

Present Value of a Level Perpetuity

- Example 6.6 What is the present value of $600 perpetuity at 7% discount rate?

$$PV = \frac{PMT}{i}$$

- PV = $600 ÷ .07 = $8,571.43

Checkpoint 6.4

The Present Value of a Level Perpetuity

What is the present value of a perpetuity of $500 paid annually discounted back to the present at 8 percent?

Checkpoint 6.4

STEP 1: Picture the problem

With a perpetuity, a timeline doesn't have an ending point but goes on forever, with the same cash flow occurring period after period, or in this case, year after year:



STEP 2: Decide on a solution strategy

Because calculating the present value of a perpetuity only involves simple division, we don't need to look at an Excel solution or any unique keystrokes with a financial calculator; instead, using Equation (6–5), we just divide the amount you receive at the end of each period forever by the interest rate.

Checkpoint 6.4

STEP 3: Solve

Substituting $PMT = \$500$ and $i = .08$ into Equation (6–5), we find

$$PV = \frac{\$500}{.08} = \$6{,}250$$

Thus, the present value of this perpetuity is $6,250.

STEP 4: Analyze

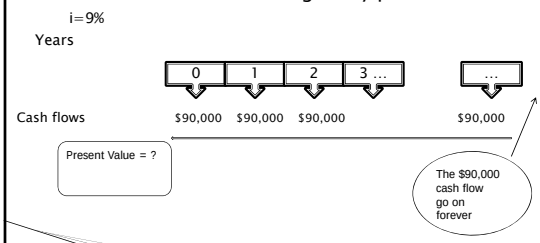
Notice there is no symbol for the future value of a perpetuity in this equation. This is because there isn't a future time period when things end since a perpetuity goes on indefinitely. So, how much will this perpetuity be worth at the end of 2 years or 100 years? The answer is $6,250. That is because this perpetuity will always return $500—regardless of what the time period is, the present value of a perpetuity paying $500 at 8 percent is always $6,250.

Checkpoint 6.4: *Check Yourself*

What is the present value of stream of payments equal to $90,000 paid annually and discounted back to the present at 9 percent?

Step 1: Picture the Problem

- With a level perpetuity, a timeline goes on forever with the same cash flow occurring every period.



Step 2: Decide on a Solution Strategy

- Present Value of Perpetuity can be solved easily using mathematical equation as given by equation 6-5.

Step 3: Solve

$$PV = \frac{PMT}{i}$$

- PV = $90,000 ÷ .09 = **$1,000,000**

Step 4: Analyze

- Here the present value of perpetuity is $1,000,000.
- The present value of perpetuity is not affected by time. Thus, the perpetuity will be worth $1,000,000 at 5 years and at 100 years.

Present Value of a Growing Perpetuity

- In growing perpetuities, the periodic cash flows grow at a constant rate each period.
- The present value of a growing perpetuity can be calculated using a simple mathematical equation.

Present Value of a Growing Perpetuity

$$PV = \frac{PMT_{period\ 1}}{i - g}$$

- PV = Present value of a growing perpetuity
- $PMT_{period\ 1}$ = Payment made at the end of first period
- i = rate of interest used to discount the growing perpetuity's cash flows
- g = the rate of growth in the payment of cash flows from period to period

Checkpoint 6.5

The Present Value of a Growing Perpetuity

What is the present value of a perpetuity stream of cash flows that pays $500 at the end of year one but grows at a rate of 4% per year indefinitely? The rate of interest used to discount the cash flows is 8%.

Checkpoint 6.5

STEP 1: Picture the problem

With a growing perpetuity, a timeline doesn't have an ending point, but goes on forever, with the cash flow growing at a constant rate period after period, or in this case, year after year:



STEP 2: Decide on a solution strategy

Because calculating the present value of a growing perpetuity only involves substituting into Equation (6–6), we don't need to look at an Excel solution, or any unique keystrokes with a financial calculator. Instead, we just divide the amount you receive at the end of each period forever by the interest rate minus the growth rate.

Checkpoint 6.5

STEP 3: Solve

Substituting $PMT_{period\ 1}$ = \$500, g = .04, and i = .08 into Equation (6–6), we find

$$PV = \frac{PMT_{period\ 1}}{i - g} = \frac{\$500}{.08 - .04} = \$12{,}500$$

Thus, the present value of the growing perpetuity is \$12,500.

STEP 4: Analyze

Comparing the value of the \$500 level perpetuity in Checkpoint 6-5 to the \$500 perpetuity that grows at 4% per year we see that adding growth to the cash flows has a dramatic effect on its value. To see why this occurs, consider the year 50 payment under both the level perpetuity and growing perpetuity. For the level perpetuity, this payment is still \$500; however, for the growing perpetuity the payment for year 50 is the following:

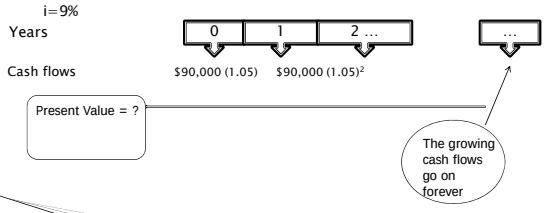
$$PMT_{year\ 50} = \$500(1 + .04)^{50} = \$3{,}553.34$$

Checkpoint 6.5: *Check Yourself*

What is the present value of a stream of payments where the year 1 payment is \$90,000 and the future payments grow at a rate of 5% per year? The interest rate used to discount the payments is 9%.

Step 1: Picture the Problem

- With a growing perpetuity, a timeline goes on for ever with the growing cash flow occurring every period.



Step 2: Decide on a Solution Strategy

- The present value of a growing perpetuity can be computed by using equation 6-6.
- We can substitute the values of PMT ($90,000), i (9%) and g (5%) in equation 6-6 to determine the present value.

Step 3: Solve

$$PV = \frac{PMT_{period\ 1}}{i-g}$$

- PV = $90,000 ÷ (.09-.05)
  = $90,000 ÷ .04
  = $2,250,000

Step 4: Analyze

- Comparing the present value of a level perpetuity (checkpoint 6.4: check yourself) with a growing perpetuity (checkpoint 6.5: check yourself) shows that adding a 5% growth rate has a dramatic effect on the present value of cash flows.
- The present value increases from $1,000,000 to $2,250,000.

6.3 Complex Cash Flow Streams

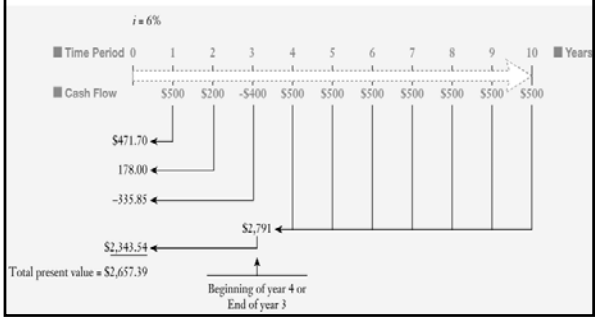
Complex Cash Flow Streams

- The cash flows streams in the business world may not always involve one type of cash flows. The cash flows may have a mixed pattern. For example, different cash flow amounts mixed in with annuities.
- For example, figure 6-4 summarizes the cash flows for Marriott.

Complex Cash Flow Streams

Figure 6.4

Present Value of Single Cash Flows and an Annuity ($ value in millions)



Complex Cash Flow Streams

- In this case, we can find the present value of the project by summing up all the individual cash flows by proceeding in four steps:
  1. Find the present value of individual cash flows in years 1, 2, and 3.
  2. Find the present value of ordinary annuity cash flow stream from years 4 through 10.
  3. Discount the present value of ordinary annuity (step 2) back three years to the present.
  4. Add present values from step 1 and step 3.

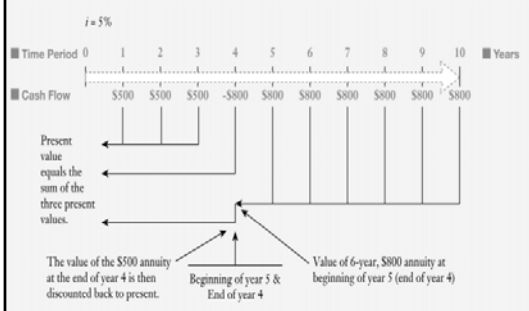
Checkpoint 6.6

The Present Value of a Complex Cash Flow Stream

What is the present value of cash flows of $500 at the end of years through 3, a cash flow of a negative $800 at the end of year 4, and cash flows of $800 at the end of years 5 through 10 if the appropriate discount rate is 5%?

Checkpoint 6.6

STEP 1: Picture the problem



Checkpoint 6.6

STEP 2: Decide on a solution strategy

This problem involves two annuities and the single (negative) cash flow. Once their present value is determined they will be added together. The $500 annuity over years 1 through 3 can be discounted directly to the present using Equation (6–2b), while the $800 cash outflow at the end of year 4 can be discounted back to present using Equation (5–2). Since it is an outflow, it will carry a negative sign and be subtracted from the present value of the inflows. To determine the present value of the 6-year, $800 annuity over years 5 through 10, we must first consolidate the 6-year annuity that runs from years 5 through 10 into an equivalent single cash flow at the beginning of year 5, which is the same as the end of year 4, using Equation (6–2b). We now have an equivalent single cash flow at the end of year 4 that we can bring directly back to present using Equation (5–2). Once everything is in today's dollars, we simply add them together.

Checkpoint 6.6

STEP 3: Solve

Using the Mathematical Formulas.

Here we have two annuities. One of them, an annuity of $500 over years 1 through 3, can be discounted directly back to the present by multiplying it by the Annuity Present Value Interest Factor or

$$\left[\frac{1 - \frac{1}{(1 + .05)^3}}{.05}\right]$$

for a value of $1,361.62. The second annuity, which is a six-year annuity of $800 per year over years 5 through 10, must be discounted twice—once to find the value of the annuity at the beginning of year 5, which is also the end of year 4, and then again to bring that value back to the present. The value of the second annuity at the end of year 4 is found by multiplying it ($800) by the Annuity Present Value Interest Factor, or

$$\left[\frac{1 - \frac{1}{(1 + .05)^6}}{.05}\right]$$

resulting in a value $4,060.55. In effect, we have now consolidated the annuity into an equivalent single cash flow at the end of year 4. This equivalent single cash flow is then discounted back to the present by multiplying it by the value of $1/(1.05)^4$ for a value of $3,340.62. Since cash flows in the same time period can be added and subtracted from each other, to arrive at the total present value of this investment, we subtract the present value of the $800 cash outflow at the end of year 4 (which is $658.16) from the sum of the present value of the two annuities ($1,361.62 and $3,340.62). Thus, the present value of this series of cash flows is $4,044.08. Remember, once the cash flows from an investment have been brought back to present, they can be combined by adding and subtracting to determine the project's total present value.

Step 3 cont.

Checkpoint 6.6

Using a Financial Calculator. Using a financial calculator, we can arrive at the same answer:

(a) The present value of the first annuity, years 1 through 3 (give it a positive sign because it is an inflow) = **$1,361.62**

	N	I/Y	PV	PMT	FV
Enter	3	5		500	0
Solve for			-1,361.62		

(b) The present value of the $800 cash **outflow** (thus it is given a negative sign because it is an outflow) = **−$658.16**

	N	I/Y	PV	PMT	FV
Enter	4	5		0	-800
Solve for			658.16		

Step 3 cont.

Checkpoint 6.6

(c) (Part 1) The value at end of year 4 of the second annuity (which is a 6-year annuity), years 5 through 10 (give it a positive sign because it is an inflow) = $4,060.55,

	N	I/Y	PV	PMT	FV
Enter	6	5		800	0
Solve for			-4,060.55		

and then,

(Part 2) The present value (give it a positive sign because it is an inflow) of the $4,060.55 (which was calculated in c: part 1 above and is received at the end of year 4) = **$3,340.62**.

	N	I/Y	PV	PMT	FV
Enter	4	5		0	4,060.55
Solve for			-3,340.62		

(d) Summing the present values together, the total present value = **$4,044.08**.

Using an Excel Spreadsheet. Using Excel, the cash flows are brought back to the present using the =PV function, keeping in mind that inflows will take on positive signs and outflows negative signs.

Checkpoint 6.6

STEP 4: Analyze

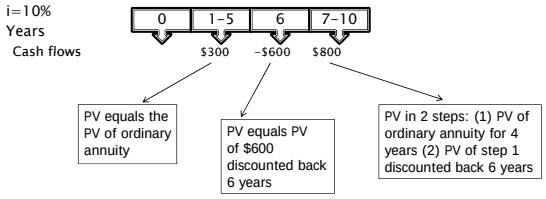
When cash flow from different time periods is expressed in the same time period's dollars, they can be added in the case of an inflow, or subtracted it in the case of an outflow—to come up with a total value at some point in time. In fact, we will combine **Principle 1: Money Has a Time Value** with **Principle 3: Cash Flows Are the Source of Value** later in the book to value stocks, bonds, and investment proposals. The bottom line is that understanding the time value of money is a key to making good decisions.

Checkpoint 6.6: *Check Yourself*

What is the present value of cash flows of:
- $300 at the end of years 1 through 5,
- −$600 at the end of year 6,
- $800 at the end of years 7–10

if the appropriate discount rate is 10%?

Step 1: Picture the Problem



Step 2: Decide on a Solution Strategy

- This problem involves two annuities
  - (years 1–5, years 7–10) and
  - single negative cash flow in year 6.
- The $300 annuity can be discounted directly to the present using equation 6–2b.
- The $600 cash outflow can be discounted directly to the present using equation 5–2.

Step 2: Decide on a Solution Strategy

- The $800 annuity will have to be solved in two stages:
  - Determine the present value of ordinary annuity for four years.
  - Discount the single cash flow (obtained from the previous step) back 6 years to the present using equation 5-2.

Step 3: Solve

- Using the Mathematical Formula
- (Step 1) PV of $300 ordinary annuity

$$\text{Present Value} = PMT\left[\frac{1-\frac{1}{(1+i)^n}}{i}\right]$$

Step 3: Solve

- PV = $300 {[1-(1/(1.10)^5] ÷ (.10)}
  = $300 {[0.379] ÷ (.10)}
  = $300 {3.79)
  = **$ 1,137.24**

Step 3: Solve

- Step (2) PV of -$600 at the end of year 6
- $PV = FV \div (1+i)^n$
- $PV = -\$600 \div (1.1)^6$
  $= \$338.68$

Step 3: Solve

- Step (3): PV of $800 in years 7-10
- First, find PV of ordinary annuity of $800 for 4 years.
- $PV = \$800 \{\left[1-(1/(1.10)^4\right] \div (.10)\}$
  $= \$800 \{[.317] \div (.10)\}$
  $= \$800 \{3.17)$
  $= \$2,535.89$

Step 3: Solve

- Second, find the present value of $2,536 discounted back 6 years at 10%.
- $PV = FV \div (1+i)^n$
- $PV = \$2,536 \div (1.1)^6$
  $= \$1431.44$

Step 3: Solve

- Present value of complex cash flow stream
  = sum of step (1) + step (2) + step (3)
  = $1,137.24 – $338.68 + $1,431.44
  = **$2,229.82**

Step 3: Solve

- Using a Financial Calculator

	Step 1	Step 2	Step 3 (part A)	Step 3 (Part B)
N	5	6	4	6
1/Y	10	10	10	10
PV	$1,137.23	$338.68	$2,535.89	$1,431.44
PMT	300	0	800	0
FV	0	-600	0	2535.89

Step 4: Analyze

- This example illustrates that a complex cash flow stream can be analyzed using the same mathematical formulas.
- If cash flows are brought to the same time period, they can be added or subtracted to find the total value of cash flow at that time period.

Figure 6.2

Timeline of a Five-Year, $500 Annuity Discounted Back to the Present at 6 Percent

To find the present value of an annuity, discount each cash flow back to the present separately and then add them. In this example, we simply add up the present value of five future cash flows of $500 each to find a present value of $2,106.18.

